XI. Spin Angular Momentum or Simply "Spin"

- Focus on Electron's Spin Angular Momentum

  [but "spin" can be assigned to particles other than electron]

- Why "Spin"?
  - Need it to understand periodic table [B, C, N, O, F, Ne]

    e.g. p orbitals ($\ell = 1$, $m_\ell = 1, 0, -1 \Rightarrow 3$ states)

    but periodic feature among elements suggested

    p orbitals have $3 \times \underbrace{2}_{\text{from spin}} = 6$ states

  - Early experiments asked for it, in addition to orbital AM
  - Need it to explain high-precision hydrogen spectrum

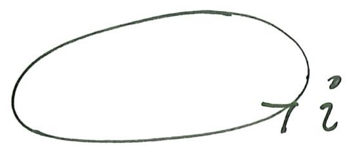
A. Concept on Experimental Measurement of Angular Momentum

- Discussed Orbital AM $\quad L^2 \to l(l+1)\hbar^2, \quad l = 0, 1, 2, \ldots$

  $L_z \to m_l \hbar, \quad m_l = \underbrace{l, l-1, \ldots, -l}_{(2l+1)\text{ values}}$

- How to do experiments on measuring Orbital AM?
  - Classical picture

orbiting electron (charge $-e$) $\Rightarrow$



current loop

$\Downarrow$

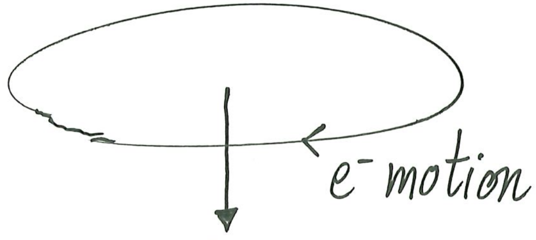
$\vec{\mu}$ = magnetic dipole moment $= i\vec{A}$

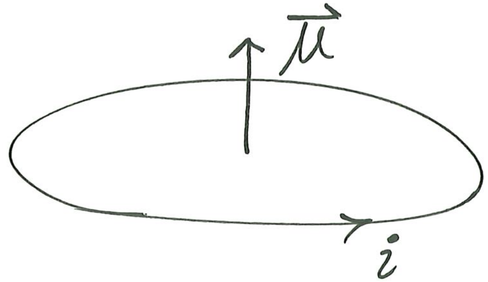
vector quantity

$\vec{A}$ → Magnitude = Area of loop

$\vec{A}$ → direction: $\perp$ plane of loop

Thus,





$(\vec{r} \times \vec{p}) \rightarrow \vec{L}$ = Orbital AM

($\because$ electron is negatively charged)

$\therefore$ Expected $\boxed{\vec{\mu}_L \propto -\vec{L}}$ (1)

Magnetic dipole moment due to orbital AM (subscript "L")

Current $\rightarrow$ $|i| = e\,\dfrac{v}{2\pi r}$ [charge passing by a point per unit time]

$$|\vec{\mu}_L| = |i| \cdot \pi r^2 = \frac{1}{2} e v r$$

Angular Momentum $\rightarrow$ $|L| = r\, m_e v$

$$\Rightarrow \frac{|\vec{\mu}_L|}{|L|} = \frac{e}{2m_e} \quad (2)$$

Called Gyromagnetic Ratio

Putting (1) & (2) together:

for an electron $\rightarrow$ $\boxed{\vec{\mu}_L = \dfrac{-e}{2m_e}\vec{L}}$ (3)

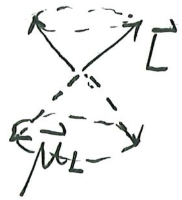
This is "Think Classical"

"Go Quantum"

$$\boxed{\hat{\vec{\mu}}_L = -\underbrace{\frac{e}{2m_e}}_{\text{a constant}}\hat{\vec{L}}} \quad (4)$$

Operator

Vector Model



opposite directions

eigenvalues of $|\vec{\mu}_L|$

Copy results: $|\vec{\mu}_L| = \dfrac{e}{2m_e}|\vec{L}| = \underbrace{\dfrac{e\hbar}{2m_e}}_{\text{a combination of } \overbrace{e,\hbar,m_e}^{\text{constants}}}\sqrt{\ell(\ell+1)}$

$$= \mu_B\sqrt{\ell(\ell+1)} \quad (5)$$

$$\mu_B \equiv \frac{e\hbar}{2m_e} = \text{Bohr Magneton} = \underbrace{\begin{cases} 9.274\times10^{-24}\ \text{J/Tesla} \\ 5.79\times10^{-5}\ \text{eV/Tesla}\end{cases}}_{\text{[Note order of magnitude \& units]}}$$

[This is the size of magnetic dipole moment expected of an atom (due to electrons)]

$(\mu_L)_z$ = z-component $= \dfrac{-e}{2m_e}L_z$ takes on $\dfrac{-e}{2m_e}m_\ell\hbar = -\mu_B \cdot m_\ell$

- But QM deals with Energy (starts with Hamiltonian)
  - Given that $\vec{L} \to \vec{\mu}_L$, how to form an energy term?

$$\boxed{\vec{\mu}_L \text{ and magnetic field } \vec{B} \text{ interact}}$$

$$\boxed{U_{magnetic} = -\vec{\mu}_L \cdot \vec{B}} \quad (6)$$

in an external applied $\vec{B}$

an additional term in $\hat{H}$

- Recall: Atom (spherically symmetric $U(r)$)
  $\Rightarrow$ No idea about what x, y, z directions are about!
- Applied $\vec{B}$ field $\Rightarrow$ $\vec{B}$ <u>defines</u> $\underbrace{\text{a direction}}_{\text{call it } z\text{-direction}}$

i.e. $\vec{B} = B\hat{z}$ (no loss of generality) for a state of quantum # m_ℓ

$$\therefore\ U_{magnetic} = -\vec{\mu}_L \cdot B = -(\mu_L)_z B = +\frac{e\hbar}{2m_e} B\, m_\ell = \underbrace{\mu_B \cdot B}\cdot m_\ell$$

[$B \sim$ few Tesla, $\mu_B \sim 10^{-5}$ eV/Tesla, $U_{magnetic} \sim 10^{-4}-10^{-5}$ eV <u>tiny but detectable</u>]

<u>Take-Home Message</u>

- electron has charge $(-e) \rightarrow \vec{L}$ leads to $\vec{\mu}_L = -\frac{e}{2m_e}\vec{L}$
- $\vec{\mu}_L$ interacts with $\vec{B} \Rightarrow U_{magnetic} = -\vec{\mu}_L \cdot \vec{B} = -(\mu_L)_z B$
- To do experiments on orbital AM, play with applied $\vec{B}$ field (7)
- $\vec{B} = 0$ / $\vec{B} \neq 0$

$\vec{B}=0$ (p-orbitals)		$\vec{B}\neq 0$ (p-orbitals)	m_ℓ
Y_{11}, Y_{10}, Y_{1-1} (degenerate)	— — — $E_{B=0}$	— $(E_{B=0} + \mu_B B)$	$+1$
		— $E_{B=0}$	0
		— $(E_{B=0} - \mu_B B)$	-1

"Removed the degeneracy"

- This is the physics behind the $\underbrace{\text{Zeeman Effect}}_{}$,

one spectral line ($\vec{B}=0$) splits into several ($\vec{B}\neq 0$)
(1902 Nobel Prize)

- Note that $\underbrace{m_\ell = -1}_{}$ has lowest energy

$\vec{\mu}_L$ tends to be "aligned" with $\vec{B}$ $\left((\mu_L)_z = -\mu_B m_\ell = +\mu_B\right)$
and thus $\vec{L}$ tends to be "anti-aligned" with $\vec{B}$ $(L_z = m_\ell \hbar = -\hbar)$

- Important for discussion on Spin

s-orbital (e.g. H-atom in ground state)
$\Rightarrow \ell = 0 \Rightarrow m_\ell = 0$ only
$\Rightarrow$ Don't expect to observe any effect in $\vec{B}$-field
<u>if</u> there is <u>only</u> orbital AM

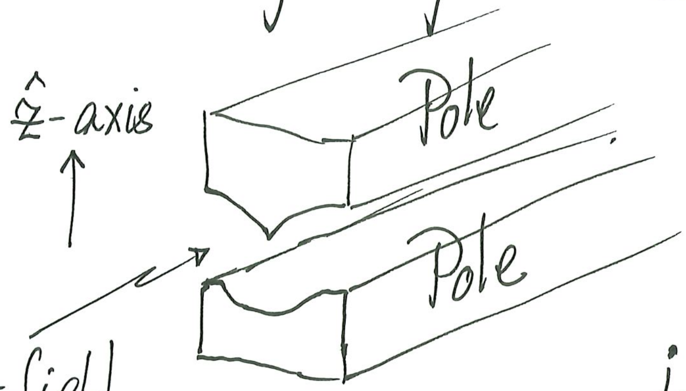
(8)

But this is <u>NOT</u> what experiments showed! (See Sec. C, Stern-Gerlach Expt)

B. Inhomogeneous Magnetic Field exerts force on magnetic dipole moment

- An example of inhomogeneous (non-uniform) magnetic field

  [Uniform field : $\vec{B}(x,y,z) = \vec{B}$ (same $\vec{B}$ at different places)]



← Special designed magnet for giving inhomogeneous field

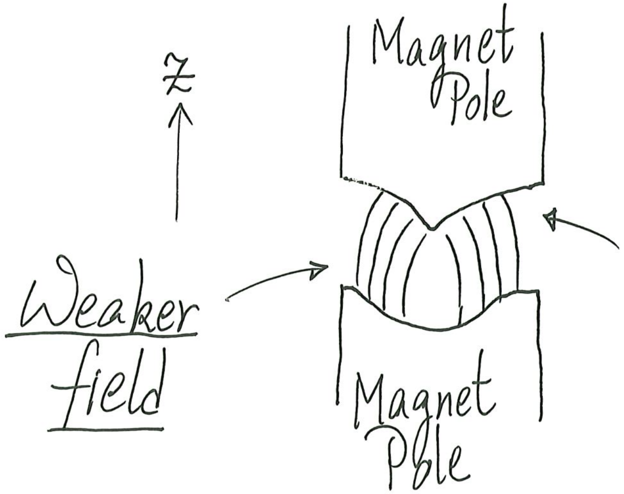
$\vec{B}$-field is inhomogeneous inside

i.e. $\vec{B}(\vec{r})$

- So what?

  more than a torque

  - A magnetic dipole moment $\vec{\mu}$ feels a force in an inhomogeneous $\vec{B}$-field



Inhomogeneous $\vec{B}$-field

[denser field lines $\Rightarrow$ stronger field]

$B(z) \Rightarrow$ Non-uniform

(B is different at different places)

$\therefore$ there is a $\dfrac{dB}{dz} \neq 0$

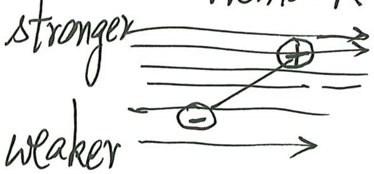
$$\vec{F} = \text{Force}^{+} \text{ on } \vec{\mu} \text{ due to inhomogeneous } \vec{B}\text{-field}$$

$$= -\vec{\nabla}(-\vec{\mu}\cdot\vec{B}) \qquad (-\vec{\nabla}(\text{potential energy}) = \text{force})$$

$$= (\vec{\mu}\cdot\vec{\nabla})\vec{B}$$

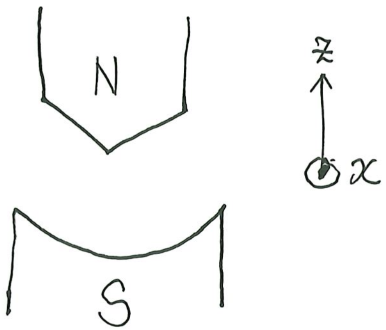
$\therefore$ As long as $\vec{B}$ varies, there is a force on $\vec{\mu}$.

$^{+}$Remark: There is an electric dipole moment in an inhomogeneous $\vec{E}$-field analogy.



net force on $\vec{p}_{electric} \sim (\vec{p}_{el.}\cdot\vec{\nabla})\vec{E}$

Key Point



As atom travels down the inhomogeneous field (into the page), it experiences a force

$$F_z = \underbrace{\mu_z}\ \underbrace{\left(\frac{\partial B}{\partial z}\right)}$$

Note: $\frac{\partial B_z}{\partial z}$ more precisely

F_z: pushing atom up or down

$\left(\frac{\partial B}{\partial z}\right)$: control by design of magnet (a constant)

μ_z: key quantity

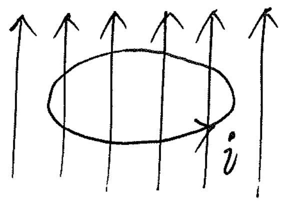
$\boxed{\text{z-component of Magnetic dipole moment}}$

along $+\hat{z}$ direction $\Rightarrow$ F_z pushes atom up

along $-\hat{z}$ direction $\Rightarrow$ F_z pushes atom down

[if atom carries a magnetic dipole moment]

Classical picture: $\vec{\mu} \approx$ current loop



in Uniform field

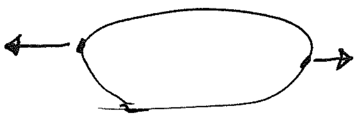
[Recall: $\underbrace{(-e)\vec{v}}_{\sim \vec{J}} \times \vec{B}$ = force on moving charge]

$\sim \vec{J} \times \vec{B}$

current density

Non-uniform Field

$\frac{\partial B}{\partial z} \neq 0$

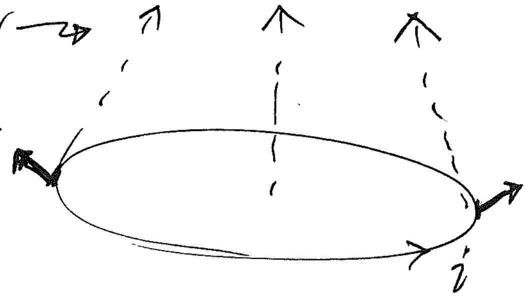
$-\vec{F}$  $\vec{F}$

cancel

(no net force)

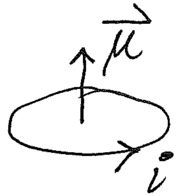
stronger

weaker



Forces only cancel in the plane of loop

But Net Force $F_z \neq 0$

$\Rightarrow$ Net Force in direction of $\frac{\partial B}{\partial z}$ for 

For 

Net Force in opposite direction

$\therefore F_z \propto \mu_z \frac{\partial B}{\partial z} \Rightarrow$ <u>different μ_z experiences different forces</u>

Example: Hydrogen atoms (all excited to 2p states)

- Only knowledge of orbital AM ($\ell = 1$, $m_\ell = 1, 0, -1$)

$L_z = \hbar, 0, -\hbar$

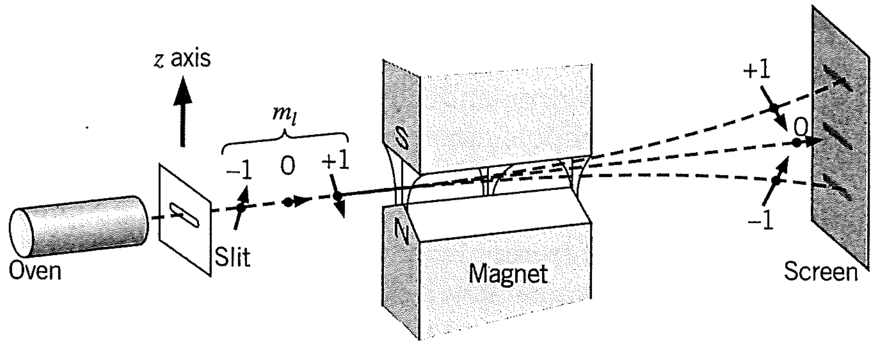
- Passing atoms through inhomogeneous $\vec{B}$-field [assuming all stay in 2p]

$(\mu_L)_z = -\mu_B, 0, +\mu_B \quad (m_\ell \cdot \mu_B)$

No force

opposite forces

We would expect to see



Schematic diagram of Stern-Gerlach experiment. A beam of atoms from an oven passes through a slit and then enters a region where there is a nonuniform magnetic field. Atoms with their magnetic dipole moments in opposite directions experience forces in opposite directions.

[Taken from Krane, "Modern Physics"]

Key Point

- Different μ_z: different forces
- This experimental set up is called the Stern-Gerlach Experiment (SG)

Remarks:

- Ideas
  - electric dipole moment experiences a force in inhomogeneous electric field
    (e.g. atoms seeking strong field locations, polarized spheres seeking strong field locations)
  - magnetic dipole moment experiences a force in inhomogeneous magnetic field

are at the heart of many modern physics experiments!
Optical tweezers, Optical lattice (forming controllable artificial crystals), electrorheological/magnetorheological fluids